20071005

Addendum to last week:

○ Parameter $\mathfrak{z}\left(e^{-t}\right) = e^{-iNg_s} = e^{N\varepsilon_2} = e^{-2\pi i N/k+N}$

↑ GW, inst. counting ; counting degree c_2 ; ↑ CS

○ Perturbative v.s. nonperturbative

The parameters $\underset{\substack{\| \\ i\varepsilon_2}}{\overset{\text{inst.}}{-i\varepsilon_1}} = \overset{\text{GW}}{g_s} = \overset{\text{CS}}{\frac{2\pi}{k+N}}$ k: level, N: rank

In GW invariant : g_s is a formal variable

$$\sum_{\substack{d\geq 1\\ g\geq 0}} \mathfrak{z}^d g_s^{2g-2} C(g,d) \in \mathfrak{z}\, g_s^{-2}\,\mathbb{Q}[[\mathfrak{z}, g_s^2]]$$

$g_s \fallingdotseq 0$ perturbative

In instanton counting :

$$H_T^*(pt) \cong \mathbb{C}[\varepsilon_1, \varepsilon_2] \underset{\text{localization}}{\rightsquigarrow} \mathbb{C}(\varepsilon_1, \varepsilon_2)$$

"generic point"

All genus GW inv's are packed in a rational function !

nonperturb.

In CS:

we will explain the expansion w.r.t. $g_s = \frac{2\pi}{k+N}$

(the CS perturbation theory) (In fact $\frac{2\pi}{k}$)

In quantum group approach to JW inv. (Reshetikhin-Turaev), the quantum group is specialised at $\sqrt[\ell]{1}$,

where $\ell = 2(k+N)$. (ℓ^{th} root of 1)

"nonperturb.?"

Thus $e^{\varepsilon_1} = e^{ig_s} = e^{\frac{2\pi i}{k+N}} = e^{\frac{4\pi i}{\ell}} = \left(\sqrt[\ell]{1}\right)^{\frac{1}{2}}$!

○ Jones-Witten invariants (or Chern-Simons theory)

<u>Ref</u>: Ohtsuki Quantum invariants, App. F

M^3 : 3-mfd, cpt, oriented

G : compact Lie group e.g. $SU(N)$

simple for simplicity

A : G-connection on the trivial b'dle $M\times G$

1-form with value in $\mathfrak{g}$

$\mathcal{A}$ = the space of G-connections $=\Omega^1(M;\mathfrak{g})$

$$CS(A)=\frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$$

"Chern-Simons functional"

(where "Tr" & "xy" ($x, y\in\mathfrak{g}$) must be understood appropriately.

This is a function (or lagrangian) on $\mathcal{A}$. But is very different from the usual one in the physics, as it is independent of the Riemannian metric.

<u>Recall</u> $\dfrac{\delta CS}{\delta A}=0 \iff A$: flat connection i.e. $F_A=0$

$\therefore A \leftrightarrow$ "rep." $\pi_1(M)\longrightarrow G$

$\mathcal{G}$ = the group of bundle automorphism
= Map(M, G) (gauge group in math.)

$\mathcal{G} \curvearrowright \mathcal{A}$ by gauge transformation

$$g^*A = g^{-1}dg + gAg$$

$\mathcal{A}/\mathcal{G}$ = the space of $\mathcal{G}$-orbits of G-connections
"the space of fields"

NB. As is usual for a quotient space, it is important to consider how we should consider $\mathcal{A}/\mathcal{G}$ as a space, as $\mathcal{G}$ has stabilizers in general.

Exercise CS is not a function on $\mathcal{A}/\mathcal{G}$, but

$$CS : \mathcal{A}/\mathcal{G} \longrightarrow \mathbb{R}/\mathbb{Z}$$ is well-defined.

We now define Jones-Witten invariant by "quantizing" the Chern-Simons functional:

$k \in \mathbb{Z}_{>0}$ (level)

$$Z_k(M) \equiv Z_{k,G}(M) = \int_{\mathcal{A}/\mathcal{G}} DA \, \exp(2\pi i k \, CS(A))$$

This is very beautiful formula except that we do not know how to define the path integral.

○ Incorporation of a link

L : link $= \coprod_i L_i$ (component)

$\mathrm{Hol}_{L_i}(A)$ = the holonomy of A along L_i
$\in$ conjugacy class of G

R_i : finite dimensional representation of G

$$\Rightarrow \quad \mathrm{tr}_{R_i} \mathrm{Hol}_{L_i}(A) =: W_{R_i}^{L_i}(A)$$

(Wilson line observable)

$$Z_{k,G,R_1,\ldots,R_\ell}(M,L) = \int_{\mathcal{A}/\mathcal{G}} DA \, \exp(2\pi i k\, CS(A)) \prod_{i=1}^{\ell} W_{R_i}^{L_i}(A)$$

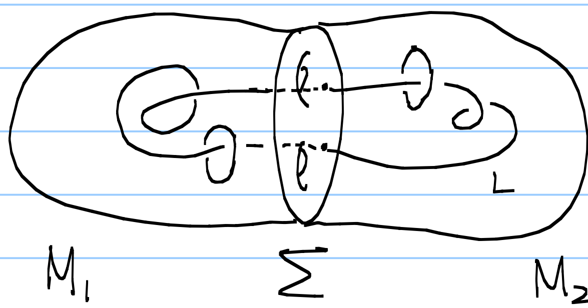
(Jones-Witten link invariant)

This is a "correlation" function in the (quantum) Chern-Simons theory.

Later I will explain the perturbative expansion of the JW invariant, which is relevant for the large N duality.

But we start with "hamiltonian approach" to the quantization problem. (topological quantum field theory)
oriented

We cut M along a surface ($= 2\dim C^\infty$-mfd) Σ



more precisely we should have in mind
$\left(\begin{array}{l}\partial M_1 = \Sigma \\ \partial M_2 = -\Sigma\end{array}\right.$

$\mathcal{A}_\Sigma$ = space of G connection on $\Sigma \times G$

ψ
a

$\mathcal{A}^i_a$ = space of G-connections A on M_i
s.t. $A|_\Sigma = a$

$\curvearrowleft \mathrm{Ker}(\mathcal{G}^i \to \mathcal{G}_\Sigma)$

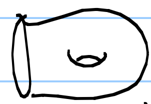
grp of b'dle automorphisms on $\Sigma \times G$

We expect $Z_k(M) =$

$$= \int_{\mathcal{A}_\Sigma / \mathcal{G}_\Sigma} Da \int_{\mathcal{A}^1_a / \mathrm{Ker}(\mathcal{G}^1 \to \mathcal{G}_\Sigma)} DA^1 e^{2\pi i k CS(A^1)} \int_{\mathcal{A}^2_a / \mathrm{Ker}(\mathcal{G}^2 \to \mathcal{G}_\Sigma)} DA^2 e^{2\pi i k CS(A^2)}$$

☆ $a \mapsto \displaystyle\int_{\mathcal{A}^i_a/\mathrm{Ker}} DA^i \, e^{2\pi\sqrt{-1}k\, CS(A^i)}$: a "function" on $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$

But this is <u>not</u> quite correct.



$g \in \mathcal{G}_\Sigma$ $\tilde{g}$ its extension to M^i

$a \in \mathcal{A}_\Sigma$ A^i "

$CS(\tilde{g}^* A^i) - CS(A^i) =: C(a, g)$ depends only on a, g
(Wess-Zumino term)

Then $e^{2\pi i C(a,g)}$ defines a line b'dle $\mathcal{L}$ on $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$.

So ☆ : a section of the line b'dle $\mathcal{L}^{\otimes k}$ on $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$.

$Z(\Sigma)$ = the "Hilbert" space of such sections

$Z(-\Sigma) = Z(\Sigma)^*$

$Z(M_1) \in Z(\Sigma)$, $Z(M_2) \in Z(\Sigma)^*$

$$Z(M) = \langle Z(M_1) \mid Z(M_2) \rangle$$

(Atiyah's topological quantum field theory)

But $Z(\Sigma)$ = the space of all sections is too large.
As is common in quantization, we should pick up a smaller space* by choosing a "polarization".

Pick a complex structure J on Σ.
Then
$$\mathcal{A}_\Sigma \cong \Omega^1(\Sigma, \mathfrak{g})$$
$$\cong \Omega^{0,1}(\Sigma, \mathfrak{g}\otimes\mathbb{C}) \leftarrow (\infty\text{-dim'l}) \text{ cpx mfd}$$

Moreover $\mathcal{G}_\Sigma^{\mathbb{C}} = \mathrm{Map}(\Sigma, G^{\mathbb{C}})$: cpxification of $\mathcal{G}_\Sigma$
acts on $\mathcal{A}_\Sigma$ holomorphically by
$$A^{0,1} \mapsto g^{-1}\bar{\partial}g + g^{-1}A^{0,1}g$$
Also $\mathcal{L}$ has a natural holo. structure.

Then it is natural to put
$$Z(\Sigma) = \text{space of holomorphic sections of } \mathcal{L}^{\otimes k} \text{ on } \mathcal{A}_\Sigma/\mathcal{G}_\Sigma^{\mathbb{C}}$$

Comment:
We could also consider an intermediate vector space of sections of the symplectic quotient
$\mu^{-1}(0)/\mathcal{G}_\Sigma$ = moduli of flat connections on Σ.
The above is its geometric quantization.

Rem $\mathcal{A}_\Sigma / \mathcal{G}^{\mathbb{C}}_\Sigma$ = moduli stack of $G^{\mathbb{C}}$-bundles on (Σ, J)

So $H^0(\mathcal{L}^{\otimes k})$ = the space of conformal blocks!

When $L \subset M$, $\{p_1, \dots, p_\ell\} = L \cap \Sigma$

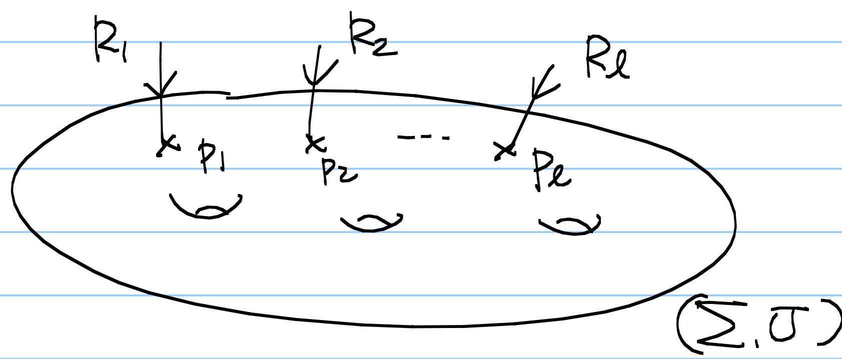
$Z_{G, R_1, \dots, R_\ell}(\Sigma, p_1, \dots, p_\ell)$

= the space of holo. sections of the line b'dle moduli stack of parabolic $G^{\mathbb{C}}$-b'dles on (Σ, J)

i.e, $G^{\mathbb{C}}$-b'dle together with reduction of $G^{\mathbb{C}} \to$ Borel at each marked pt

The line b'dle $\mathcal{L}$ now depends also on R_i

= the space of conformal blocks attached to the data



There is one thing to be checked:

$Z(\Sigma)$ "should" be a topological invariant.

$\exists$ (projectively) flat connection on the bundle of conformal blocks over the moduli space of pointed Riemann surfaces.

○ Perturbation theory

Suppose A is a flat connection:

$$CS(A+\alpha) = CS(A) + \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\alpha \wedge d_A\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha)$$

– stationary phase approximation (ignore the cubic term)

$$Z_k(M) \underset{k:\text{large}}{\sim} \sum_{[A]:\text{flat connection}} \alpha([A])\, e^{2\pi i k\, CS(A)}$$

Rem. In general, the moduli space of flat connections are not isolated points, not even a smooth manifold.

But we ignore this point, and assume $H^i_A = 0$.

Recall $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$

By analytic continuation $\displaystyle\int_{-\infty}^{\infty} e^{i\lambda x^2}\frac{dx}{\sqrt{\pi}} = \frac{1}{\sqrt{|\lambda|}}\exp\left(\frac{\pi i}{4}\operatorname{sgn}\lambda\right)$

$\lambda \in \mathbb{R}$

For a quadratic form Q on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} e^{ikQ(x)}\,\frac{dx_1\cdots dx_n}{\pi^{n/2}} = \frac{1}{\sqrt{|\det kQ|}}\exp\left(\frac{\pi i}{4}\operatorname{sgn} Q\right)$$

In our case we want to apply this formula to

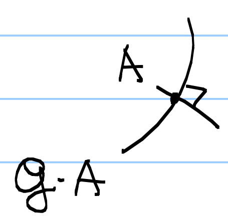
$$\mathbb{R}^n \rightsquigarrow T_{[A]}(\mathcal{A}/\mathcal{G})$$

$$Q \rightsquigarrow \frac{1}{4\pi}\int_M \mathrm{Tr}(\alpha\wedge d_A\alpha)$$

∘ $T_{[A]}(\mathcal{A}/\mathcal{G})$:

We take a "slice" to the gauge group orbit

standard recipe :



Pick up a <u>Riemannian metric</u> g on M^3, and consider

$$\mathrm{Ker}(d_A^* : \Omega^1(M)\otimes\mathfrak{g} \to \Omega^0(M)\otimes\mathfrak{g})$$

$$\cong T_{[A]}(\mathcal{A}/\mathcal{G})$$

We also need to understand the Jacobian of
$\operatorname{Ker} d_A^* \xrightarrow[\cong]{} \mathcal{A}_\Sigma / \mathcal{G}_\Sigma$ to compare the Feynman measure.

Then finally (see [Atiyah] for more details) $\det Q$, $\operatorname{sgn} Q$ are expressed in terms of $\Delta_A^{(i)}$: laplacian on $\Omega^i(M;\mathfrak{g})$
& $D_A = (d_A + * d_A *) : \Omega^{odd} \to \Omega^{even} \overset{*}{\cong} \Omega^{odd}$

Answer

- $$\frac{1}{\sqrt{\det k Q}} = \frac{\sqrt{\det \Delta_A^{(0)}}}{\left(\dfrac{\det k^2 \Delta_A^{(1)}}{\det k^2 \Delta_A^{(0)}} \right)^{1/4}} \leftarrow \text{Jacobian}$$

- $\operatorname{sgn} Q = \operatorname{sgn} D_A$

Now we use the Ray-Singer ζ-function regularization to define $\det \Delta_A^{(i)}$, $\operatorname{sgn} D_A$

$$\zeta_A(s) = \operatorname{tr} \Delta_A^{-s} = \sum_{\lambda \neq 0} \lambda^{-s} \qquad \lambda : \text{eigenvalue of } \Delta_A^{(i)}$$

$$\zeta_A(0) = \text{"} \dim \Omega^i \text{"} = 0 \qquad \text{in odd dim i.e. in our case}$$

$$\exp(-\zeta_A'(0)) = \text{"}\det\text{"} \Delta_A^i$$

& η-inv : $\eta_A(s) = \sum_{\lambda \neq 0} |\lambda|^{-s} \operatorname{sgn} \lambda$ $\quad \lambda$: eigenvalue of D_A $\in \mathbb{R}$

$$\eta_A(0) = \text{“}\operatorname{sgn}\text{”}\, D_A$$

Th (Cheeger, Müller)

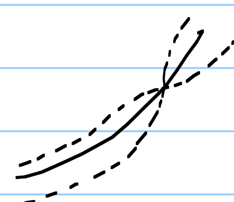
$$\frac{(\det \Delta_A^{(0)})^{3/2}}{(\det \Delta_A^{(1)})^{1/2}} = \text{Reidemeister torsion}$$

metric independent!

The phase factor is more subtle, as $\eta_A(0)$ is not a topological invariant.

The invariant depends on the choice of the "framing" of M, i.e., $\underbrace{TM \cong M \times \mathbb{R}^3}_{\text{trivialization}}$

framing of link



string

$\rightarrow$ ribbon

○ Perturbation theory

- finite dim'l model

$$Z_k := \int_{\mathbb{R}^n} dx \ \exp[ik(Q(x)+T(x))]$$ T: cubic form ("T_{abc})

$$= \int_{\mathbb{R}^n} dx \ \exp(ikQ(x)) \sum_{m=0}^{\infty} \frac{1}{m!} (ikT(x))^m$$

We introduce a new variable $u \in \mathbb{R}^n$ (auxiliary field)

$$\frac{\partial}{\partial u_a} \exp(i\langle u,x\rangle)\Big|_{u=0} = ix_a$$

$$\frac{\partial^2}{\partial u_a \partial u_b} \exp(i\langle u,x\rangle)\Big|_{u=0} = ix_a \, ix_b \ , \ \cdots\cdots$$

$$\therefore T(x)^m = \left(\sum_{a,b,c=1}^{n} T_{abc} x_a x_b x_c\right)^m$$

$$= \frac{1}{i^{3m}} \left(\sum T_{abc} \frac{\partial}{\partial u_a}\frac{\partial}{\partial u_b}\frac{\partial}{\partial u_c}\right)^m \exp(i\langle u,x\rangle)\Big|_{u=0}$$

We put $\exp(i\langle u,x\rangle)$ to $\exp(ikQ(x))$

$$\exp(ikQ(x) + i\langle u,x\rangle)$$

Complete the square ;

$$\exp\left[ikQ(x') - \frac{i}{4k}\langle u, Q^{-1}u\rangle\right] \qquad x' = x + \frac{1}{2k}Q^{-1}u$$

Our measure is translation invariant.
(the same is true for the Feynman measure)

$$\therefore Z_k = k^{-\frac{N}{2}} \frac{1}{\sqrt{\det Q}} \exp\left(\frac{i\pi}{4} \operatorname{sgn} Q\right) \qquad \text{quadratic in } u$$

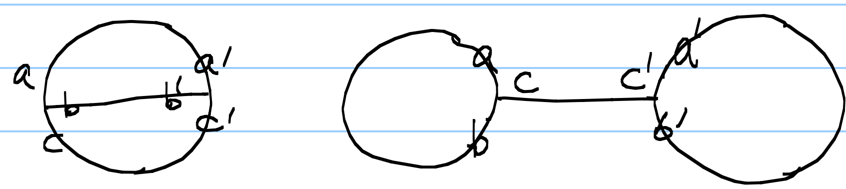
$$\times \sum_{m=0}^{\infty} \frac{(-1)^m k^m}{m!} \underbrace{\left(\sum_{a,b,c} T_{abc} \frac{\partial}{\partial u_a} \frac{\partial}{\partial u_b} \frac{\partial}{\partial u_c} \right)^m}_{\text{Cubic in } u} \underbrace{\exp\left(-\frac{i}{4k} \langle u, Q^{-1} u \rangle\right)}_{\text{II}} \Bigg|_{u=0}$$

We can expand the second part : $\sum \frac{1}{n!} \left(-\frac{i}{4k}\right)^n \langle u, Q^{-1} u \rangle^n$ ← quadratic in u
(NB, term $= 0$ unless $3m = 2n$)

simplest term
$m=2$, $n=3$

$T_{abc} T_{a'b'c'} Q^{aa'} Q^{bb'} Q^{cc'}$ & $T_{abc} T_{a'b'c'} Q^{ab} Q^{a'b'} Q^{cc'}$
(& permutation of indices)

graphically



We put T at vertex. $m = \#$ vertex
Q edge $n = \#$ edge
$m - n = e(\Gamma)$

(Feynman graph)

We get $\exp\Big(\sum_{\Gamma:\text{ connected graph}} k^{-e(\Gamma)} \frac{Z(\Gamma)}{|\mathrm{Aut}\,\Gamma|} \Big)$.

$Z(\Gamma)$ is defined as above

We apply this argument to the ∞-dim'l setting:

We represent $Q^{-1}\alpha = \int_M L(\cdot, x)\, \alpha(x)$

$\mathfrak{g} \otimes \mathfrak{g}$-valued

i.e. $Q^{-1} = L *$

$\rightsquigarrow$ $Z(\Gamma)$ is given by an integration over $\underbrace{M \times \cdots \times M}_{2n}$

<u>Remark</u>, k is shifted by $k + h^\vee$

dual Coxeter #

(quantum correction)

$$\Big(\eta_A(0) - \eta_{\mathrm{triv.}}(0) \propto h^\vee \cdot CS(A) \Big)$$

Comment:

The pertubative invariants can be proved to be independent of a Riemannian metric. They give contributions of a flat connection (or a component of the moduli space of flat connections).

However, the exact invariant is well-defined only for an integer k, it is probably not possible to single out the contribution of a flat connection for a general 3-manifold.

Q. Do you understand why link invariants in S^3 can be defined for arbitrary param. not necessarily roots of unity?

A. No, except by computation.